

Analyzing the Population Growth Equation in the Solow Growth Model Including the Population Frequency: Case Study: USA

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Abstract

To obtain a high rate of economic growth needs the recognition of the factors and the potential facilities. A change in the population growth rate could have a significant impact on the economic growth. On one hand with a change in population growth rate would affect the working-age and the manpower, on the other, the life cycle, consumption and as a result the saving and investment would be influenced. The Solow economic growth model considers the population growth (labor force growth) as a function that has a constant growth rate. In this model the equilibrium long-term per capita volume depends on the population growth, in a sense that with an increase in the population growth, per capita capital, per capita production and per capita consumption would decrease. This happens when the economic strength increases with the population growth in a steady-state period. The initial classic assumption is changed in this article based on the available frequencies in the population growth equation that depend on Four known economic (Kitchen Cycle, Juglar Cycle, Kuznets Cycle, Kondratieff Wave) cycles. For this purpose the Fourier series approach that is applied in most of engineering mathematics is adapted here. In order to find the appropriate sine and cosine waves statement of the Fourier series fit for the population growth equation, first the population curve is fitted on the an exponential function, then the residuals is fitted on the Fourier series.. In this model we obtain the similar results of the economic growth model of Solow.

Key words: Solow growth model, Population growth, The Fourier series, Frequency

1. Introduction

In growth theory it is usually assumed that population growth follows an exponential law. The Solow growth model assumes that labor force grows exponentially. This is clearly unrealistic because, exponential growth implies that population increases to infinity as time tends to infinity. On the other hands, generalized logistic equation utilizes three hypotheses about human population growth: (1) when population size is small, growth is exponential; (2) population is bounded; and (3) the rate of population growth decreases to zero as time tends toward infinity after making this substitution, the generalized Solow model is then solved in closed form, demonstrating that the intrinsic rate of population growth does not influence the long-run equilibrium level of capital per worker. Then the process described by the model is asymptotically stable, small changes on the initial condition of the growth process do not affect the long run performance. when population variables the more realistic conditions, the intrinsic rate of population growth plays no role in determining the long run equilibrium levels of per capita consumption, capital and output, while with exponential population growth an increase in the intrinsic rate of population growth leads to lower levels of these variables. It also shows that equilibrium per capita levels of consumption, capital and output are greater than those of the classical model decreases.

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A variety of growth curves have been developed to model both unpredated, intraspecific population dynamics and more general biological growth. Most successful predictive models are shown to be based on extended forms of the classical Verhulst logistic growth equation.

P.F. Verhulst(1838) the famous French researcher who worked on the population biology tried to improve the mathematical curving approaches that contained the above mentioned features regarding population growth. From empirical studies he found out that for every static biological population, proportional growth rate would fall in general since under pure natural conditions that absolute population growth rate would lead to deficiency in per capita subsistence resources; hence; a reduction in population. Therefore Verhulst assumed that the proportional population growth rate is a reducing function of the population.

He applied a simple form of the general Ricatti differential equation:

$$\frac{dy}{dx} = ax'' + by^2$$

In order to assess the population growth as follows:

$$\frac{dp}{dt} = rP(1 - kP)$$

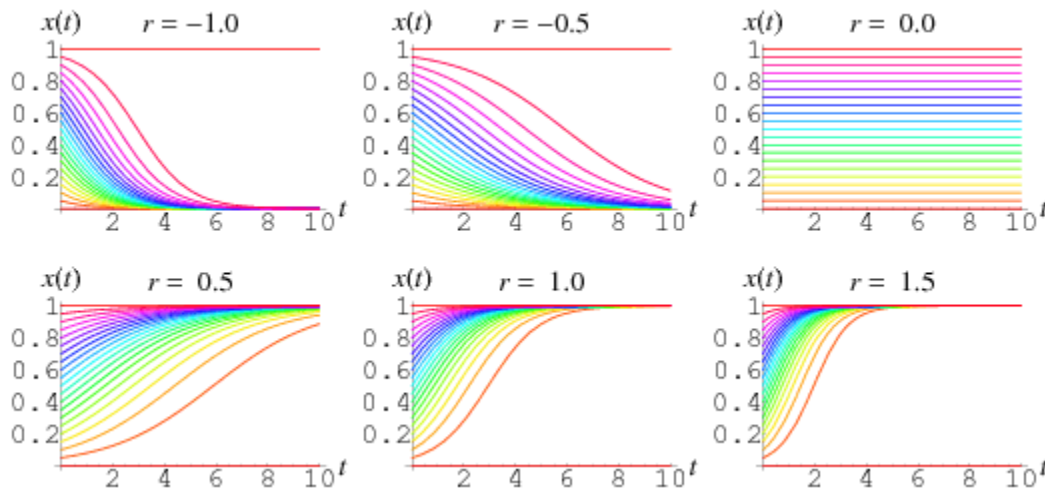
Now if we solve the differential equation above, we would have:

$$P = \frac{L}{1 + e^{r(B-t)}}$$

This value is known as the logistic function of population growth.

For different rates of r the following diagrams could be observed.

Figure1



It is obvious that the population cure will never achieve the tangential values. This issue (even after considering woman as a biological kind) corresponds with the fact that the population of no country could reach zero or infinity.

In the continuation the logistic growth models would be analyzed and a general logistic growth model will be introduced engulfing all the previous models.

2. Analysis of logistic Growth Models

2.1. Von Bertalanffy's growth equation

In all logistic growth models the Carrying- Capacity is presented by K.VonBertalanffy found and presented a different growth equation through which he modified the Verhulst's curve in a manner that with developed Bernoulli differential equation and was able to present the conditions that Verhulst had exerted on the logistic curve and presented his equation as follows:

$$\frac{dN}{dt} = rN^{\frac{2}{3}} \left[1 - \left(\frac{N}{K} \right)^{\frac{1}{3}} \right]$$

The equation is solved as follows:

$$N(t) = K \left[1 + \left[1 - \left(\frac{N_0}{K} \right)^{\frac{1}{3}} \right] e^{\frac{1}{3}rtK^{\frac{1}{3}}} \right]^3$$

2.2. Richards growth equation

Richard (1959) suggested his proposed equation as follows:

$$\frac{dN}{dt} = rN \left[1 - \left(\frac{N}{K} \right)^{\beta} \right]$$

Now if we solve this Bernoulli differential equation the following will result:

$$N(t) = K \left[1 - e^{\beta rt} \left[1 - \left(\frac{N_0}{K} \right)^{-\beta} \right] \right]^{\frac{1}{\beta}}$$

2.3. Blumberg’s equation

Blumberg (1968) improvised the Verhulst growth equation. He noticed that the main restriction in the logistic tangent is its turning point. He considered the inherent growth rate function as a function independent of time in a sense that this rate would eventually reach a zero growth rate, hence calling this curve an ultra logistic curve, presented as follows:

$$\frac{dN}{dt} = rN^{\alpha} \left[1 - \frac{N}{K} \right]^{\gamma}$$

2.4. Gompertz growth function

The Gompertz logistic growth is presented as follows:

$$\frac{dN}{dt} = \frac{r}{\beta^{\gamma}} N \left[1 - \left(\frac{N}{K} \right)^{\beta} \right]^{\gamma}$$

After solving this Bernoulli differential equation we get:

$$N(t) = K \exp \left\{ \left[\ln \left(\frac{N_0}{K} \right) \right]^{1-\gamma} + \frac{r}{K\beta^{\gamma}} (-1)^{\gamma} (1-\gamma)t \right\}^{\frac{1}{1-\gamma}}$$

Of course when $\beta \rightarrow 0$, through the following mathematical simplification we get:

$$\lim_{\beta \rightarrow 0} \frac{K^{\beta} - N^{\beta}}{\beta} = \lim_{\beta \rightarrow 0} \frac{e^{\beta \ln K} - e^{\beta \ln N}}{\beta} = \lim_{\beta \rightarrow 0} \frac{\sum_{n=0}^{\infty} \frac{(\beta \ln K)^n}{n!} - \sum_{n=0}^{\infty} \frac{(\beta \ln N)^n}{n!}}{\beta} = \ln \left(\frac{K}{N} \right)$$

This Bernoulli differential equation is simplified as follows:

$$\frac{dN}{dt} = rN \left[\ln \left(\frac{K}{N} \right) \right]^{\gamma}$$

2.5 The Generic Growth Function

Toner et al (1976) improved the Verhulst equation as follows:

$$\frac{dN}{dt} = rN^{1+\beta(1-\gamma)} \left[1 - \left(\frac{N}{K} \right)^{\beta} \right]^{\gamma}$$

Now if we solve the differential equation above, we would have:

$$N(t) = \frac{K}{1 + \left[(\gamma - 1)\beta r K^{\beta(1-\gamma)} t + \left[\left(\frac{K}{N_0} \right)^{\beta} - 1 \right]^{1-\gamma} \right]^{\frac{1}{1-\gamma}}}$$

3. Generalized logistic growth function

Now, after having reviewed the logistic equation we emphasize on the general logistic equation where each one of the above mentioned equations present a specific function in this newly introduced equation and the Solo-Swan model is being analyzed based on the following equation:

$$\frac{dN}{dt} = rN^{\alpha} \left[1 - \left(\frac{N}{K} \right)^{\beta} \right]^{\gamma}$$

By considering any one of these parameters we can obtain the above mentioned equations.

Now the features of generalized logistic function would be expressed:

- 1-When the population size is less ($N \sim 0$) in relation to the Carrying- Capacity, the labor force growth with a constant increasing rate.
- 2-When the population size is more ($N \sim 0$) in relation to the carrying capacity, the labor force growth with the population growth rate is reduced due to the restricted economic resources.
- 3-In distance time horizons ($N=K$) the population growth rate feeds to swing towards zero.

The logistic growth models are not able to observe some of the realities, i.e. in countries like The U.S, France, Germany, Canada, Spain, Greece... the population growth rate is not leaning towards zero. In order to explain this phenomenon which contains some frequencies in the time length, the following formula could be presented:

Population growth rate=British-Death+ Immigration-Emigration

Any change in these variables naturally would change the population growth rate. Even if the British rate is negative, the population growth rate could be positive. In general, any economic variable influencing the above mentioned Four variables would affect the Population growth rate.

The effect of population transfer on the age groups structure regarding time would expand and changes in age group would have its statistical significance on the economic growth, development, and reception to a certain degree. Of course the age group aspect is built-in the population transfer phenomenon.

Bo, M. Sommestad (2000) has divided the aspect of age group transfer in the Four following sub-group:

- 1- Childhood: population growth percentage under 15 years of age
- 2- Youth: population growth percentage between ages 15-29
- 3- Mideage: population growth percentage between ages 30-65
- 4- Elder: population growth percentage over 65

Therefore, by considering the fact that in our lifetime we are in one of the above age sub-groupings and its advance from group to group is inevitable, the population growth percentage can divert from its slow pace or seas and or vice versa.

Then, in general the logistic growth models are not capable of explaining the changes in population growth. The attempt is made here to express a new model based on mathematical reasoning and economic rational in a sense that first, through a higher per capital investment or at least even level it would reach Solow's growth models and second, it would be free of the Carrying- Capacity restriction saturation level and finally population growth rate would affect the per capita values in the long term.

4. Economic rational of model

4.1 Long Waves, Long Cycles, and Long Swings

Have there been, since (say) 1800, "long wave" cycles of some 45-60 years in world rates of economic growth? If so, we are in a downswing. Some 60 years' data have accumulated since Kondratieff and Schumpeter's works, with much improvement in data for earlier periods and in econometric sophistication. Yet Solomou [1987] shows that what looks like a regular long wave in twentieth century growth rates can be "dismantled" into a number of

episodic traverses without obvious cyclical character. Reijnders [1990], on the other hand, shows that for the nineteenth century, where no long wave is evident, one appears once allowance is made for perspectivistic distortion-the influence on the data of fluctuations of longer and shorter periods than the one looked for This controversy shows that empirical analysis cannot be conducted in isolation from theory.

The only cycle of longer period than "long waves" that might be expected to affect economic growth is the "long cycle" in international relations [Wallerstein 1983; Modelski 1987]. This is a cycle of leadership or hegemony of some four generations-100- 120 years. During the first generation, a global war takes place, ending with the victory of a new hegemon, or world power. This state thenceforth largely controls world trade and economic relations. In Modelski's scheme, the next generation is one of uncontested leadership, the third of delegitimation, and the fourth of deconcentration, with a serious challenger emerging; then global war returns. During our two centuries, the hegemons have been Britain (till 1914) and the United States (since 1945); the global wars have been the Revolutionary/Napoleonic (1792-1815) and the 1914-45 periods. Periods of global war presumably tend to slow growth. Maddison [1982] and Solomou [1987] show that the world economy did grow decidedly more slowly through 1914-45 than in the 30 years before, and grew far faster in the 30 years afterward. Likewise, we know that growth was slow from 1792 to 1815 in the United States and Britain, and for the decade afterward also in continental Europe [Tylecote 1993a]. The 1914-45 deceleration was less pronounced for Britain, the ex-hegemon, than for any other major economy, as was the subsequent acceleration for the United States, the next hegemon. Hegemonic responsibilities seem a burden.² (One might also expect a state newly arriving at world power status to be enjoying a surge of economic dynamism at the time.) Moreover, neither the old nor the new hegemon suffered nearly as much physical destruction or social disruption as most other contestants in the global war. Much the same appears to be true for the previous global war, except for the oddity that the "retiring" and new hegemon were one and the same, Britain. This explains why Reijnders [1990], examining only Britain, found no "long cycle" output fluctuation. For the world economy, we can assume a "long cycle" fluctuation of some 120 years, involving slower growth for some 30 years around the time of "global war."

4.2. Population Cycles

While the movement of population shows unimportant and irregular fluctuations which seem to be mere reflections of economic, sanitary and other conditions, we find quite clearly great waves, the main cause of which are the great wars. The deficit of births during a war and the surplus of births in the immediate postwar period repeat themselves about thirty-three years later, when the new generations are at their time of highest fertility. For the same reason thirty-three years later a third wave occurs. Of course these subsequent waves become broader and broader, flatter and flatter, and after a hundred years entirely interfere with each other, so that it may roughly be said that a war calls forth a set of three full cycles of a hundred years duration in all. Two things usually happen to these cycles: they are either reinforced or disturbed by cycles resulting from later wars. For the first possibility Sweden is an excellent example. There it happened again and again that at the end of a cycle a new great war renewed the waves, with the result that there was a continuous succession.

These are the outlines of my explanation of the business cycle. I am, however, well aware of the fact that there are limitations to its validity. First, the argument has been intentionally simplified by not mentioning the wellknown, tho in my opinion only supplementary, role played by the rate

4.3. Demographic Cycles and Economic Growth

Usually the descending economic growth pattern that repeat itself in a contentious manner in three month periods indicates Economic Recession and as it elongates it ends up in Economic Repression. Economic stagnation is an expression referred to the periods when the economic growth is slow but not negative for a while.

In general there exist four main acknowledge economic cycles:

- 1- Kitchen Cycle with 3.3 years duration
- 2- Juglar Cycle with 9 years duration
- 3- Kuznets Cycle with 18 years duration
- 4- Kondratieff cycle with 54 years duration

The experience has been characterized, however, by repetitive fluctuations in the levels and growth rates of aggregate production and its components. The length of the shorter of these fluctuations, the business cycle,

ranges from the 40- to 45-month inventory cycle to the so-called Juglar of seven to ten years.¹ Two other classes of interruptions in the secular trend have also been singled out for study by economic historians. The first is the Kondratieff cycle, a movement of roughly fifty years which has been primarily identified in price series. The second is the Kuznets cycle, or "long swing," which in length is between the Juglar and the Kondratieff.

To date, the analysis of long swings has been based mainly on American experience, in which long waves in investment, output growth, migration, capital flows, and population change have ranged from fifteen to twenty-five years. It is generally agreed that demographic factors and, in particular, changes in birth and migration rates form an important element in American long swings. For example, Kuznets emphasizes the role of population growth in generating waves of "population sensitive" and "other" capital formation. The differential productivities of these two investment components in turn influence the rate of expansion of aggregate supply, and thus economic growth.

5. Mathematical reasoning of model

5.1 Fourier Series Expanded In Time t with period T

Let the function $f(t)$ be periodic with period $T = 2L$ where

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}$$

In this case, our lower limit of integration is 0.

Hence the Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) dt$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos n\omega t dt \quad b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin n\omega t dt$$

(Note: half the range of integration = π/ω)

5.2 Exponential Fourier series

An alternative, and often more convenient, approach to Fourier series is to use complex exponentials instead of sines and cosines. Indeed, Euler's formula:

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

The complex Fourier series for a (piecewise continuous) real or complex function f is:

$$f(x) = \sum_{k=-\infty}^{k=\infty} c_k e^{ikx}$$

that the complex Fourier coefficients are obtained by taking the inner products:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

with the associated complex exponential. Pay attention to the minus sign in the integrated exponential the result of taking the complex conjugate of the second argument in the inner product. It should be emphasized that the real and complex Fourier formulae are just two different ways of writing the same series! Indeed, if we apply Euler's formula to and compare with the real Fourier formulae, we find that the real and complex Fourier coefficients are related by:

$$a_k = c_k + c_{-k}$$

$$b_k = i(c_k - c_{-k})$$

Let us find the Fourier series for the exponential function . It is much easier to evaluate the integrals for the complex Fourier coefficients, and so:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-ik)x} dx = \frac{e^{(a-ik)x}}{2\pi(a-ik)} \Big|_{-\pi}^{\pi} = \frac{e^{(a-ik)\pi} - e^{-(a-ik)\pi}}{2\pi(a-ik)} = (-1)^k \frac{e^{a\pi} - e^{-a\pi}}{2\pi(a-ik)}$$

$$= \frac{(-1)^k (a + ik) \sinh a\pi}{\pi(a^2 + k^2)}.$$

Therefore , the desired Fourier series is

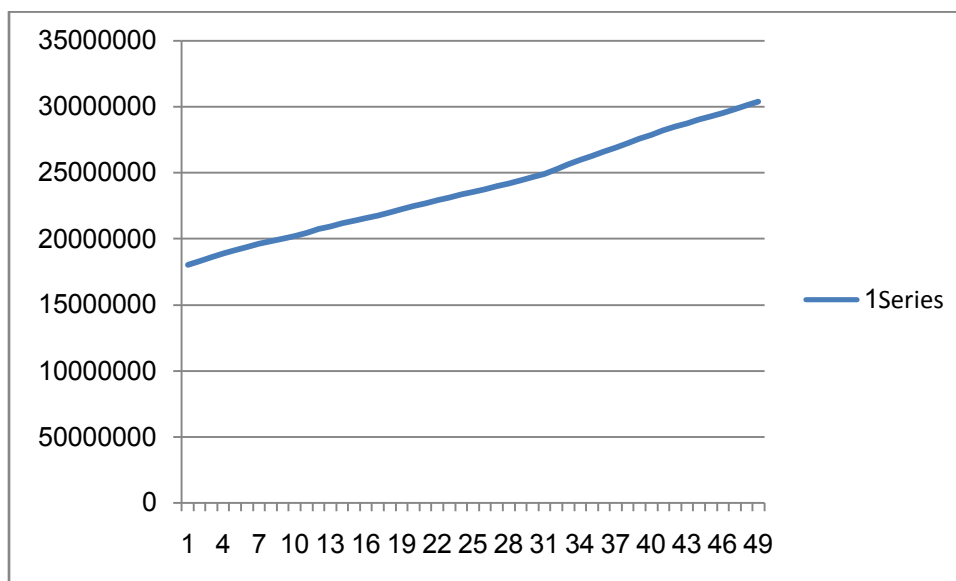
$$e^{ax} \sim \frac{\sinh a\pi}{\pi} \sum_{k=-\infty}^{k=\infty} \frac{(-1)^k (a + ik)}{(a^2 + k^2)} e^{ikx}$$

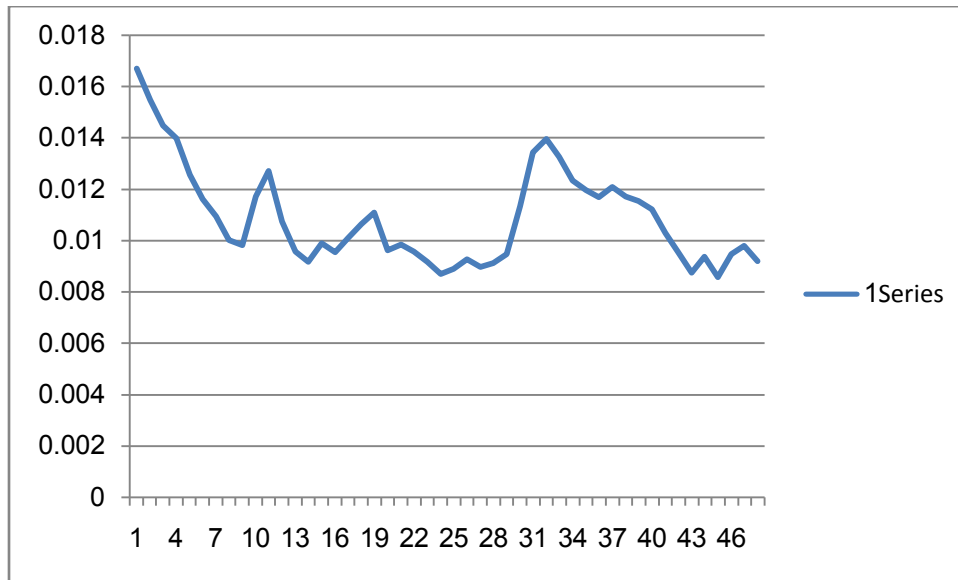
6. Analysis of The Fourier series population growth equation

In the 1920s the Russian economist Nicholas Kondratieff discovered a pattern of recurring 50-year megacycles in the economies of Germany, France, Great Britain, the United States and presocialist Russia. Each cycle is characterized by four distinct phases-(1) a growth period culminating in an inflationary peak, (2) a short-lived primary depression, (3) a plateau phase and (4) a long period of economic stagnation including a secondary depression. The U.S. has experienced three complete Kondratieff cycles-from the 1780s to 1843, 1843 to 1896 and 1896 to 1940. The first phase of each cycle has lasted an average of 27 years, only to be followed by a one-year second phase-a primary depression. The third-plateau-phase, lasting four to eight years, has typically failed to regain the levels of growth that preceded the primary depression. The fourth phase-economic stagnation-has averaged 19 years and been characterized by at least one major secondary depression. On average, recessions during the stagnant phase have lasted a year longer than recessions during the growth phase.

In order to obtain the Fourier series population growth equation the selected equation must have a series of frequencies, but since the derivatives of population growth have frequencies, then the main equation of population growth has frequencies as well; consequently a Fourier series fit for this equation could be written. The population growth main equation and its derivatives are shown in the figures below:

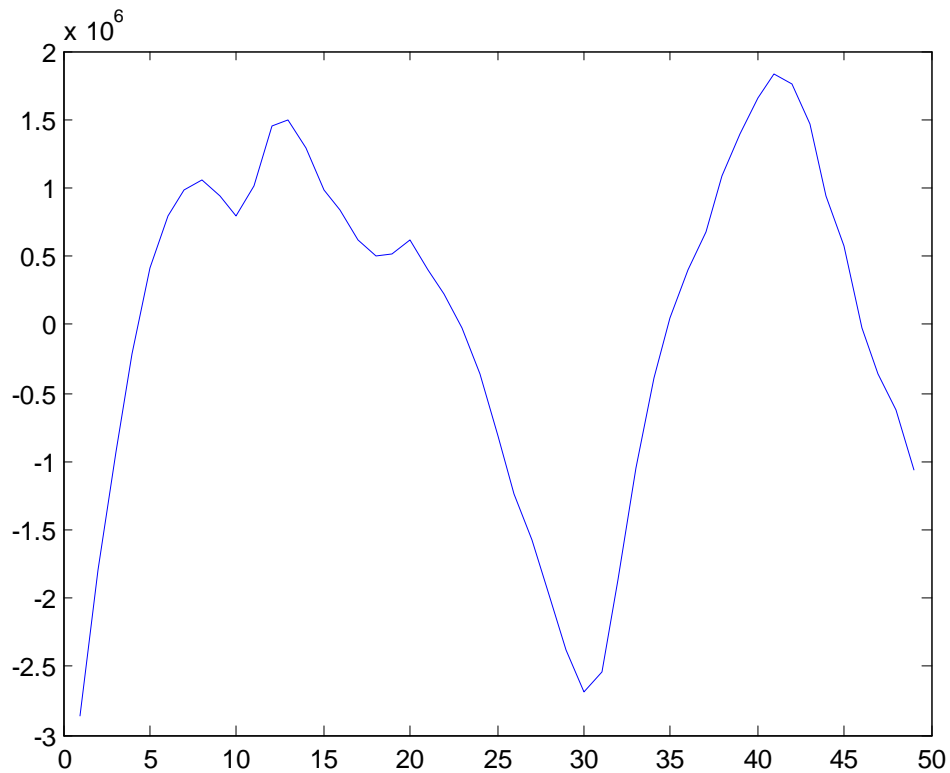
Figure2. Population and population growth U.S.A(1960-2008)





In order to find the appropriate sine and cosine waves statement of the Fourier series fit for the population growth equation, first the population curve is fitted on the an exponential function, then the residuals is fitted on the Fourier series. As can be observed in the figure bellow the residuals statements contain a series of frequencies.

Figure3. Residuals of population equation



Here of course for four reasons, four main harmonic of the function is under consideration.

- 1- The influence of statements coefficients higher than frequency (4ω) on the main function range is insignificant .
- 2- Since in very short times (high frequency) the population growth cannot have significant change, consequently, this frequencies are not presented in the population growth equation.
- 3- The population growth equation is analyzed based on the four acknowledged economic cycles: (Kitchen Cycle · Juglar Cycle · Kuznets Cycle· Kondratieff Wave)
- 4- Regarding the above mentioned changes in age structure only four age cycles could exist.

Consequently, here the following population growth equation is being suggested based on the Fourier series.

$$L(t) = a_0 + a_1 \cos(t\omega) + b_1 \sin(t\omega) + a_2 \cos(2t\omega) + b_2 \sin(2t\omega) + a_3 \cos(3t\omega) + b_3 \sin(3t\omega) + a_4 \cos(4t\omega) + b_4 \sin(4t\omega) + ae^{nt}$$

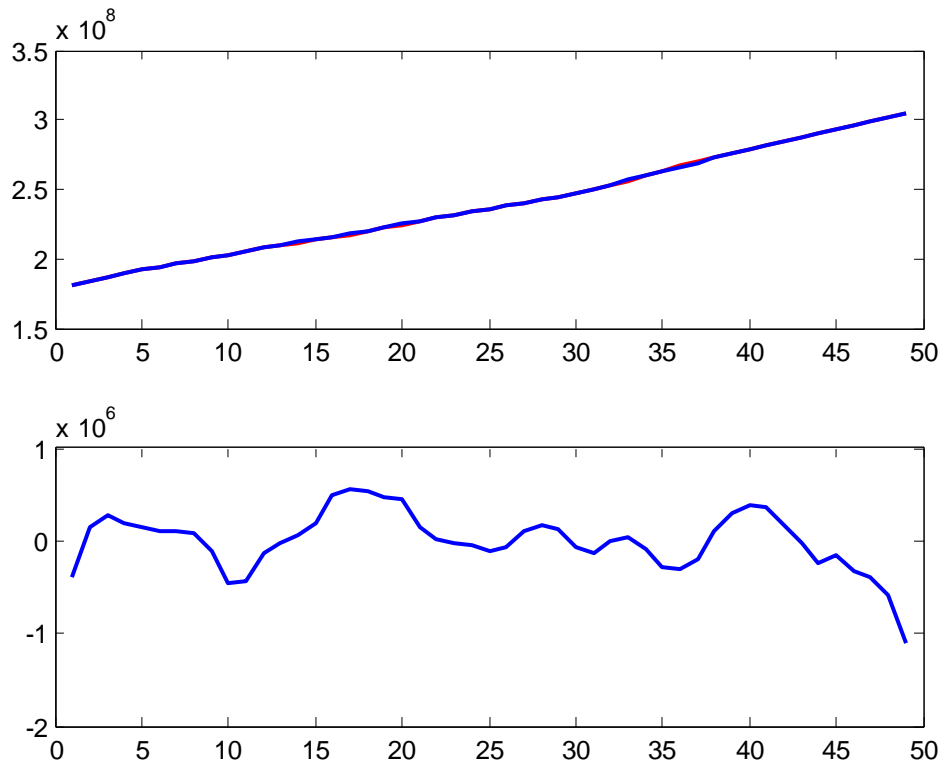
Where, ω is the popular frequency. Now by applying the advanced Matlab software and using the curve-fitting the desired values for The U.S could be obtained. The estimated parameters are as follows:

$$a_0 = -1.389e+005 \quad a_1 = -1.454e+006 \quad b_1 = 5.733e+005 \quad a_2 = -7.181e+005 \quad b_2 = -2.152e+005$$

$$a_3 = -2.759e+005 \quad b_3 = 1.779e+005 \quad a_4 = -5.093e+004 \quad b_4 = -1.652e+005 \quad \omega = 0.2103$$

$$a = .816e+008 \quad n = 0.01059$$

Figure4. the real(Blue line), estimated data(red line) and residuals



The obtained frequency for The U.S is 0.2103 which is equal to 25 years. The more the frequency value the less the time, that is, the economy is enjoying a higher power in a short period of time.

Now the population growth is obtained:

$$\frac{\dot{L}(t)}{L(t)} = \rho(t)$$

And with a mathematical simplification we would get the following:

$$\dot{L}(t) = -a_1\omega \sin(t\omega) + b_1\omega \cos(t\omega) - 2\omega a_2 \sin(2t\omega) + 2\omega b_2 \cos(2t\omega) - 3\omega a_3 \sin(3t\omega) + 3\omega b_3 \cos(3t\omega) - 4\omega a_4 \sin(4t\omega) + 4\omega b_4 \cos(4t\omega) + a n e^{nt}$$

As can be observed the population growth, just like the population variable has changeable frequencies that can be either positive or negative but in the long-term they reach a stable growth rate.

$$\lim_{t \rightarrow \infty} \rho(t) = n$$

The initial Solow-Swan equation is as follows:

$$\dot{r} = sf(r) - (\delta + n)r$$

Where, r is the per capita capital; s is the saving rate; $f(r)$ is the per capita production function; δ is the depreciation rate and n is the population growth rate.

The modified equation is presented as follows:

$$\dot{r} = sf(r) - (\delta + \rho(t))r$$

The main difference here is the population growth rate only that is no longer constant but is a function of time. Consequently we get to an advanced differential equation that is no easily solved on the time capital path and its steady-state power cannot be calculated easily.

The newly obtained differential equation from the Solo-Swan model frequency is of Bernoulli type, that is:

$$\dot{x}(t) = a(t)x + b(t)$$

Solving this equation is very complicated. Here the steady-state values of the variables are of essence. we obtain the same result as that of the Solow's economic growth. because of:

$$\lim_{t \rightarrow \infty} \rho(t) = n$$

that is:

$$\dot{r} = sf(r) - (\delta + n)r$$

7. Conclusion

The economic growth theory usually the population growth is considered as an exponential growth rate; although this seems as a non-realistic assumption. According to Solow, the existence of a positive growth rate for population, for the purpose of explaining the economic growth is essential, but once an economic system determines its population growth path based on an exogenous rate. Any increase in the population growth rate (in ratio to the previous rate) would describe lower per capita capital and production in the transition phase of the economy.

In this article we change the initial classic assumption and obtain the population growth rate based on the frequencies inherent in the population growth equation in accordance with the Four cycles. Here the Fourier series approach is adapted. To apply the Fourier series approach our equation should contain a series of frequencies, but since its derivative, that is the population growth rate has frequency therefore the main equation of population growth rate must have frequencies that can be fitted in the Fourier series and expressed. In the following figures the main equation and its derivatives are obtained. In order to find the sine and cosine statements first for the Fourier series and applicable in the population growth rate first we fit the population curve curve is fitted on the an exponential function, then the residuals is fitted on the Fourier series statement.

In this proposed model we obtain the same result as that of the Solow's economic growth.

This happens when the economic power is increased with the possibility of negative population growth rate in the steady-state.

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